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RESEARCH MEMORANDUM

EMPIRICAL BAYES POINT ESTIMATES OF TRUE SCORE USING A COMPOUND BINOMIAL ERROR MODEL

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Abstract

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EMPIRICAL BAYES POINT ESTIMATES OF TRUE SCORE USING

A COMPOUND BINOMIAL ERROR MODEL[1,2]

Jack Kearns

Educational Testing Service

Recently the empirical Bayes approach has been applied to certain problems in mental test theory. Empirical Bayes procedures are based upon the Bayes assumption of prior distributions, but utilize empirical information in lieu of making specific distributional assumptions. Several mental test theory applications have conceptualized the distribution of true scores as a prior distribution. These have included methods for the estimation of the true score distribution [Lord, 1969] and methods for obtaining point estimates of true score [Meredith, 1971; Meredith & Kearns, 1973]. Both of these approaches have used "strong" true score theory assumptions [cf. Lord & Novick, 1968, Part 6] which specify the form of the conditional distribution of observed scores for a fixed true score, i.e., the error or propensity distribution for an individual.

Point estimation methods have concentrated upon the development of estimates which are asymptotically optimal, i.e., estimates which approach a Bayes point estimator as the sample size becomes increasingly large. These estimates require essentially no *a priori* assumptions about the true score distribution (occasionally some very general assumptions are made). The Bayes point estimators are highly advantageous in that they minimize the overall expected squared error loss. This implies that the

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[1]Presented at the 1974 Spring Meeting of the Psychometric Society, Stanford University, March 28-29, 1974.

[2]I am indebted to Frederic M. Lord for a critical reading of this paper and to Dorothy T. Thayer for implementing the necessary computer programs.

Bayes point estimates are as reliable as any other type of score [cf. Meredith & Kearns, 1973, Section VI]. The speed with which the empirical Bayes point estimator will approach the Bayes point estimator will depend upon various characteristics of the propensity and true score distributions, but, in general, the samples must be reasonably large to provide estimates which are superior to the maximum likelihood estimates (observed score). An alternative approach has involved various "smoothing" procedures which have been shown to reduce overall expected loss considerably for relatively small samples, and are generally superior to the use of observed scores, but are not asymptotically optimal.

The point estimation problem has been explored for assumed Poisson and binomial error distributions. This study will extend this approach to the compound binomial error distribution used by Lord [1965, 1969] which is applicable to a fairly large class of nonspeeded, binary item tests.

Let X_N be the random variable, taking on particular values x which represent scores on an N item test. Let T be the proportion correct true score random variable with particular values τ $(0 \leq \tau \leq 1)$. If $g(\tau)$ is the distribution of true scores, and $f(x|\tau)$ is the error distribution, then the regression of T on X is given by

$$(1) \qquad \mathcal{E}(T|x) = \int_0^1 \tau h(\tau|x)d\tau = \frac{\int_0^1 \tau f(x|\tau)g(\tau)d\tau}{\int_0^1 f(x|\tau)g(\tau)d\tau} ,$$

where $h(\tau|x)$ is the "posterior" distribution of T, given x. If it is possible to calculate (1), this "posterior" mean is equivalent to the Bayes point estimator of T which minimizes the overall expected squared error loss [Maritz, 1970, p. 4].

If an individual's responses to the N items are experimentally independent, with a vector of probabilities of passing $\underline{\phi} = \{\phi_g\}$ ($g = 1, \ldots, N$), then $f(x|\tau)$ is a compound binomial distribution, where $\tau = \frac{1}{N} \sum_{g=1}^{N} \phi_g$. This distribution depends upon the vector $\underline{\phi}$ but may be expressed in terms of τ as a finite series [Lord, 1969]

$$
\text{(2)} \qquad \begin{aligned} P_N(x|\underline{\phi}) = {} & \binom{N}{x} \tau^x (1-\tau)^{N-x} + \frac{N}{2} V_2(\underline{\phi},\tau) C_2(x,\tau) \\ & + \frac{N}{3} V_3(\underline{\phi},\tau) C_3(x,\tau) + \left[\frac{N}{4} V_4(\underline{\phi},\tau) + \frac{N^2}{8} V_2^2(\underline{\phi},\tau) \right] \\ & + \left[\frac{N}{5} V_5(\phi,\tau) - \frac{5}{6} N^2 V_2(\underline{\phi},\tau) V_3(\underline{\phi},\tau) \right] C_5(x,\tau) \\ & + \ldots \text{ (N terms)} , \end{aligned}
$$

where

$$
V_r(\underline{\phi},\tau) = \frac{1}{N} \sum_{g=1}^{N} (\phi_g - \tau)^r
$$

and

$$
C_r(x,\tau) = \sum_{v=0}^{r} (-1)^{v+1} \binom{r}{v} P_{N-r}(x - v|\tau)
$$

where

$$p_N(x|\tau) = \begin{cases} \binom{N}{x} \tau^x (1 - \tau)^{N-x} & 0 \leq x \leq N \\ 0 & \text{otherwise} \end{cases}$$

Lord has studied approximations which retain only the first few terms of (2) and approximate the $V_r(\underline{\phi},\tau)$ by functions of τ, e.g.,

(3) $$V_2(\underline{\phi},\tau) \doteq V_2(\tau) = \frac{2k}{N} \tau(1 - \tau) \quad ,$$

where k is a constant which must be estimated. We shall consider here the approximation obtained by retaining only the first two terms. If solely the first term is retained, (2) reduces to the binomial error model, which must be discussed first.

THE BINOMIAL ERROR MODEL

When $f(x|\tau)$ is binomial (N,τ) [cf. Robbins, 1955], equation (1) becomes

(4) $$\mathcal{E}_N(T|X_N = x) = \frac{x + 1}{N + 1} \; \frac{p_{N+1}(x + 1)}{p_N(x)} \quad ,$$

where the values of $p_N(x)$ are the probabilities of the unconditional distribution of observed scores. With information from only N items, this regression is indeterminate [Lord & Novick, 1968, p. 514] since the distribution represented by the values of $p_{N+1}(x + 1)$ is unknown. Bayes point estimates of T are obtained by considering outcomes for only the first $N - 1$ items, i.e.,

$$(5) \qquad \mathcal{E}_{N-1}(T|X_{N-1} = x) = \frac{x+1}{N} \frac{p_N(x+1)}{p_{N-1}(x)}$$

If the items are truly equivalent, then any item may be deleted to obtain such an estimate. The substitution of sample proportions in (5) for the values of $p_N(x + 1)$ and $p_{N-1}(x)$ gives one an asymptotically optimal empirical Bayes estimator.

Meredith and Kearns [1973] have developed a procedure for assigning Bayes point estimates to individuals who obtain a given score on the N item test. This depends upon the following result for the binomial distribution [cf. Lord & Novick, 1968, p. 365, Corollary]:

$$(6) \qquad P[X_{N-1} = x - 1|X_N = x] = \frac{x}{N}$$

$$P[X_{N-1} = x|X_N = x] = \frac{N-x}{N}$$

This conditional distribution is independent of the parameter τ, and hence is valid for the entire population of individuals. If we set $X_N = x$, but treat X_{N-1} as a random variable which can assume only the two values x and $x - 1$, then the expected value of $\mathcal{E}_{N-1}(T|X_{N-1})$ conditional upon $X_N = x$ is

$$(7) \qquad \mathcal{E}_N[\mathcal{E}_{N-1}(T)|x] = \mathcal{E}_N[\mathcal{E}_{N-1}(T|X_{N-1} = x \text{ or } x-1)|X_N = x]$$

$$= \frac{x}{N}\mathcal{E}_{N-1}(T|X_{N-1} = x - 1) + \frac{N-x}{N}\mathcal{E}_{N-1}(T|X_{N-1} = x)$$

This is a population estimate which may be assigned to an individual on the basis of his score on the entire N item test. This estimate is based upon information from only N items. It is not to be identified with the regression estimate of (4) which requires information from $N + 1$ items in order to be determined. The estimate of (7) contains no more information than the regression estimates of (5). It represents a probabilistic assignment, based upon the distribution of (6), to one of two possible outcomes on an $N - 1$ item test obtained by deleting an item. Meredith and Kearns [1973] used this result to obtain the estimate

$$\bar{T}(X_N = x) = \left(\frac{1}{N}\right) \sum_{i=1}^{N} \mathcal{E}_N[\mathcal{E}_{N-1}^{(i)}(T)|x] \quad ,$$

where the superscript (i) indicates that item i has been deleted. This equation combines results from all possible item deleted subtests and reduces the overall expected loss when the $\mathcal{E}(T|X_{N-1} = x)$ are estimated empirically. This assignment procedure may be extended to outcomes on an $N + 1$ item test, i.e.,

$$(8) \qquad \mathcal{E}_{N+1}[\mathcal{E}_{N-1}(T)|x] = \mathcal{E}_{N+1}[\mathcal{E}_{N-1}(T|X_{N-1} = x,\ x - 1,\ \text{or}\ x - 2)|X_{N+1} = x]$$

$$= \frac{x}{N+1}\, \mathcal{E}_N[\mathcal{E}_{N-1}(T)|X_N = x]$$

$$+ \left(\frac{N + 1 - x}{N + 1}\right) \mathcal{E}_N[\mathcal{E}_{N-1}(T)|X_N = x] \quad .$$

Again, this estimate contains no more information than (7) or (5).

With increasing N, there is a corresponding increase in the number of moments (N) of the true score distribution which may be obtained from the observed score distribution [Lord & Novick, 1968, p. 521].

The true score distribution can never be uniquely determined, but, with increasing N, the class of possible true score distributions becomes increasingly restricted. In addition, the relative amount of information sacrificed by using the estimate of (5) decreases. In this context, the estimate of (7) may be seen as appropriate for $X_N = x$, but based upon an inadequate amount of information about the true score distribution. The amount of information needed for the regression in (4) is equivalent to that obtained with $N + 1$ items. Since the assignment procedure of (7) may be extended to Bayes estimates obtained with fewer items (as in (8)), it becomes possible to observe the way in which, for fixed $X_n = x$, these assigned estimates change as a function of increasing N. If this functional relation is sufficiently regular, it should be possible to fit a curve for several values $\leq N$ and extrapolate to $N + 1$. This extrapolated estimate should be a close approximation to the nonidentifiable estimate of (4).

THE COMPOUND BINOMIAL ERROR MODEL

The two term approximation of (2) using the additional approximation of (3) may be written, in terms of τ, as

$$(9) \qquad P_N(x|\tau) = p_N(x|\tau) + k\tau(1 - \tau) \cdot \sum_{v=0}^{2} (-1)^{v+1} \binom{2}{v} p_{N-2}(x - v|\tau)$$

Lord [1965, 1969] suggests obtaining k such that the correlation between true and observed score is equal to the square root of the Kuder-Richardson formula-20 reliability. This value of k is

$$(10) \qquad k = \frac{N^2(N-1)\sigma_\pi^2}{2[\mu_X(N-\mu_X) - \sigma_X^2 - N\sigma_\pi^2]} ,$$

where μ_X and σ_X^2 are the mean and variance of X_N, and σ_π^2 is the variance of the item difficulties. The unconditional distribution of X_N is, using (9),

$$(11) \qquad P_N(x) = \int_0^1 p_N(x|\tau)g(\tau)dt$$

$$= p_N(x) + k \sum_{v=0}^{2} (-1)^{v+1} \binom{2}{v} \frac{(x - v + 1)(N - x + v - 1)}{N(N-1)} p_N(x - v + 1) .$$

From the definition following (2), $p_{N-2}(x - v|\tau) = 0$ unless $0 \le x - v \le N - 2$. Consequently, in equation (11), $p_N(x - v + 1)$ is equal to zero under the same conditions, i.e., when $x < v$ or $x > N - 2 + v$. With the value of k obtained from (10), equation (11) represents $N + 1$ equations in the $N + 1$ unknowns, $p_N(x)$.

Let

$$E_N(T|x) = \mathcal{E}_N(T|X_N = x)$$

for the compound binomial test, and

$$\mathcal{E}_N(T|x) = \mathcal{E}_N(T|X_N = x)$$

for the hypothetical binomial error test whose distribution is represented by the values of $p_N(x)$. Then the Bayes point estimate of T, obtained by substitution of (9) into (1), is

$$(12)\quad E_N[T|x] = \frac{\int_0^1 \tau P_N(x|\tau)g(\tau)d\tau}{\int_0^1 P_N(x|\tau)g(\tau)d\tau}$$

$$= [\frac{x+1}{N+1} p_{N+1}(x+1) + \{k \sum_{v=0}^{2} (-1)^{v+1} \binom{2}{v}$$

$$\cdot \frac{(x-v+2)(x-v+1)(N-x+v-1)}{(N+1)N(N-1)} p_{N+1}(x-v+2)\}]$$

$$\cdot [p_N(x) + \{k \sum_{v=0}^{2} (-1)^{v+1} \binom{2}{v}$$

$$\cdot \frac{(x-v+1)(N-x+v-1)}{N(N-1)} p_N(x-v+1)\}]^{-1} .$$

Dividing through by $p_N(x)$ and letting

$$A(x) = 1 + 2k \frac{x(N-x)}{N(N-1)}$$

$$B(x) = k \frac{(x+1)(N-x-1)}{N(N-1)} [\frac{p_N(x+1)}{p_N(x)}]$$

$$C(x) = k \frac{(x-1)(N-x+1)}{N(N-1)} [\frac{p_N(x-1)}{p_N(x)}] ,$$

we may write (12) as

$$(13)\quad E_N[T|x] = \frac{A(x)\mathcal{E}_N(T|x) - B(x)\mathcal{E}_N(T|x+1) - C(x)\mathcal{E}_N(T|x-1)}{A(x) - B(x) - C(x)}$$

The values of $p_N(x)$ may be obtained by solution of the set of $N + 1$ equations given by (11).

The procedures outlined in the previous section may be used to estimate $\varepsilon_N(T|x)$ with the following necessary modification. The probabilities $p_{N-1}(x)$ used in (5) can be obtained, in the case of a binomial error distribution, by deleting any item and observing the distribution. Alternatively equation (6) may be used to write

$$(14) \qquad p_{N-1}(x) = \frac{N - x}{N} p_N(x) + \frac{x + 1}{N} p_N(x + 1)$$

This equation may be used in the case of the hypothetical values of $p_N(x)$ determined by (11). The extrapolated estimates of $\varepsilon_N(T|x)$ may be found by the indicated procedure and substituted into (13).

If, as is the usual case, we have only sample estimates of the $P_N(x)$, the estimation procedure must be examined with regard to sampling variability and its effect upon the overall expected squared-error loss, which is a random variable over repeated sampling [cf. Maritz, 1970]. This is the empirical Bayes situation which we shall consider next.

EMPIRICAL BAYES ESTIMATES

Simple empirical Bayes estimates may be obtained for the regression of (13) by substitution of sample proportions, $\hat{P}_N(x)$, for the $P_N(x)$ of equation (11). However, these are not likely to be the best estimates in terms of minimizing the overall expected squared error loss [cf. Maritz, 1970, pp. 17-18] unless sufficiently large samples are used.

Several empirical Bayes estimators have been proposed which can minimize this loss for small or moderate samples and are applicable to the binomial error model [Lemon & Krutchkoff, 1969; Griffin & Krutchkoff, 1971; Copas, 1972; Bennett & Martz, 1972; Meredith & Kearns, 1973]. However, while these procedures generally have desirable small-sample properties, they are not usually the best procedures for extremely large samples, i.e., they are not asymptotically optimal. However, two of these methods [Lemon & Krutchkoff, 1969; Bennett & Martz, 1972] approach asymptotic optimality as $N \to \infty$, and are approximations of (4) rather than (5). The advantage of these two smoothing procedures is therefore a function of the size of the sample relative to the value of N. A particular method may be advantageous depending upon the particular characteristics of the distribution of true scores. All of these procedures may be used to estimate $\varepsilon_N(T|x)$ from the $\hat{p}_N(x)$ obtained from (11).

An additional empirical consideration is the stability of the ratios $\left[\dfrac{p_N(x+1)}{p_N(x)}\right]$ and $\left[\dfrac{p_N(x-1)}{p_N(x)}\right]$ needed for $B(x)$ and $C(x)$. If these ratios are estimated by substitution of the values of $\hat{p}_N(x)$, they are likely to increase sampling error unless the sample is quite large. Lord [1959] has shown that the recurrence relation

$$(15) \qquad \varepsilon_N(T|x) = 1 - \frac{N - x + 1}{x} \frac{p_N(x-1)}{p_N(x)} \varepsilon_N(T|x-1)$$

holds for $x = 1, \ldots, N$ in the population. This represents N equations in $N + 1$ unknown and reflects the indeterminacy discussed in terms of equation (4). For sufficiently large samples, we should require that the estimates of $\varepsilon_N(T|x)$ satisfy the constraints represented by (15). Alternatively, equation (15) provides a method for estimating $B(x)$ and $C(x)$ if estimates of $\varepsilon_N(T|x)$ are available, i.e.,

$$(16)\qquad \frac{p_N(x+1)}{p_N(x)} = \left(\frac{N-x}{x+1}\right)\left[\frac{\varepsilon_N(T|x)}{1-\varepsilon_N(T|x+1)}\right]$$

and

$$(17)\qquad \frac{p_N(x-1)}{p_N(x)} = \left(\frac{x}{N-x+1}\right)\left[\frac{1-\varepsilon_N(T|x)}{\varepsilon_N(T|x-1)}\right]$$

APPLICATIONS

An example is taken from Meredith and Kearns [1973]. The data represent a sample of 7718 respondents on an 11-item subtest selected from items of the School and College Ability Test. The items were selected to have approximately equal item difficulties. The computed value of σ_π^2 was .000859.

Table 1 gives results for the binomial error model assumption. The assigned estimates corresponding to (7) are shown along with extensions (as in (8)) based upon information from N^* items ($N^* < N$). In addition, the extrapolated estimates of the regression function are given. The extrapolated estimates for this and the following example were obtained by fitting a quadratic curve to the "last" four points, i.e., for N^* equal to $N-1$, $N-2$, $N-3$, and $N-4$. Table 2 shows the results for the compound binomial error model using the same data. The assigned and extrapolated estimates for the hypothetical binomial error distribution obtained from (11) are shown. These values are also presented in Figure 1. The empirical Bayes estimates corresponding to (13) are

Table 1

Data from Meredith and Kearns [1973]

Binomial Model

x	$\mathcal{E}_N(T\mid x)^+$	$\mathcal{E}_N[\mathcal{E}_{N^*}(T)\mid x]$ (N*) 10	9	8	7	6	5	4
0	.29209169	.30424	.31646	.32742	.33890	.35231	.37008	.39529
1	.33791842	.33934	.34384	.35205	.36313	.37851	.39907	.42625
2	.35174337	.35939	.36991	.38062	.39510	.41267	.43408	.46057
3	.39098276	.39801	.40684	.42167	.43690	.45405	.47391	.49746
4	.42773249	.44571	.46304	.47441	.48692	.50104	.51726	.53617
5	.52764118	.52649	.52804	.53396	.54202	.55159	.56281	.57596
6	.59550599	.59311	.59220	.59525	.59895	.60359	.60927	.61616
7	.63712462	.64677	.65427	.65490	.65497	.65507	.65539	.65609
8	.73038987	.72174	.71416	.71120	.70811	.70439	.70006	.69513
9	.77094235	.77009	.76790	.76292	.75707	.75031	.74229	.73270
10	.82655946	.82122	.81501	.80853	.80098	.79204	.78132	.76822
11	.86751275	.86217	.85567	.84812	.83938	.82906	.81661	.80118

+Extrapolated

Table 2

Data from Meredith and Kearns [1973]

Compound Binomial Model

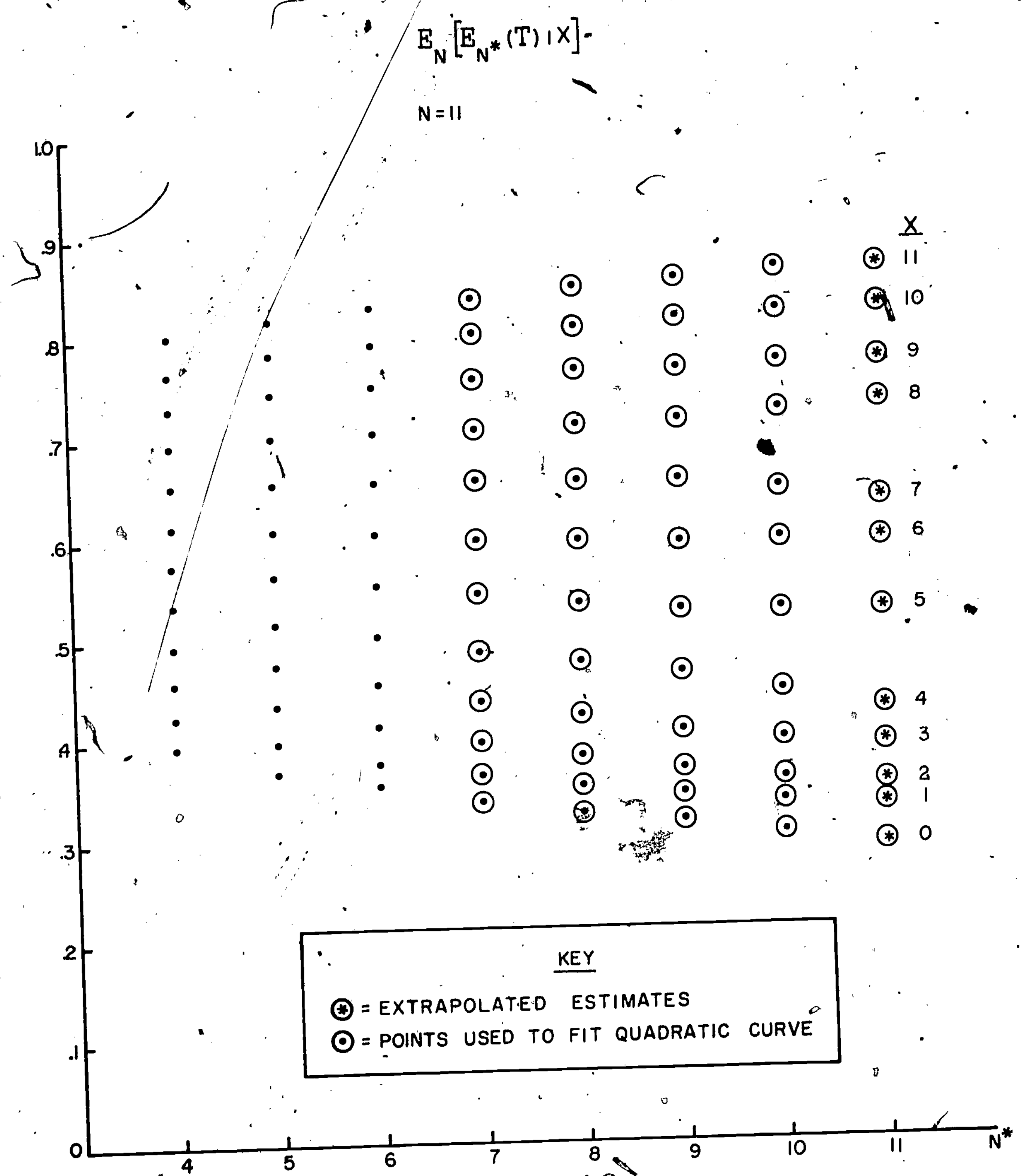
x	$E_N(T\|x)$ (a)	$E_N(T\|x)$ (b)	$\mathcal{E}_N(T\|x)^+$	$\mathcal{E}_N[\mathcal{E}_{N^*}(T)\|x]$ (N*) 10	9	8	7	6	5	4
0	.290604	.290600	.29111016	.30321	.31542	.32644	.33801	.35154	.36943	.39474
1	.336685	.336685	.33683978	.33841	.34304	.35134	.36250	.37795	.39858	.42581
2	.350897	.350892	.35125349	.35891	.36943	.38020	.39470	.41229	.43372	.46023
3	.390386	.390392	.39062644	.39775	.40667	.42145	.43667	.45380	.47365	.49720
4	.427461	.427432	.42814459	.44585	.46297	.47430	.48677	.50087	.51707	.53597
5	.527273	.527282	.52738727	.52634	.52795	.53388	.54192	.55147	.56268	.57582
6	.595268	.595272	.59523195	.59295	.59213	.59517	.59887	.60351	.60918	.61607
7	.637096	.637101	.63745358	.64688	.65422	.65484	.65492	.65502	.65535	.65605
8	.730370	.730380	.73005762	.72157	.71412	.71118	.70810	.70439	.70007	.69516
9	.771174	.771173	.77107457	.77018	.76795	.76297	.75714	.75039	.74238	.73279
10	.826972	.826973	.82671602	.82140	.81520	.80873	.80118	.79223	.78151	.76840
11	.868176	.868177	.86801046	.86264	.85611	.84854	.83977	.82943	.81695	.80147

$^+$Extrapolated

Figure 1

Data from Meredith and Kearns [1973]

Compound Binomial Model



shown for both (a) substitution of the values of $\hat{p}_N(x)$ and (b) use of (16) and (17) to estimate $\hat{B}(x)$ and $C(x)$. Since the items are so similar in difficulty, it is not surprising that these estimates correspond very closely to those in Table 1.

A second example from Lord [1965] uses one of the sixteen distributions analyzed in that study ($N = 25$, $\sigma^2_\pi = .035$, $k = 2.1$, sample size = 1000). Table 3 gives the results of applying both the compound binomial and binomial models. For all estimates there is a general lack of monotonicity which reflects the smaller sample size (and larger number of items). In addition, the estimates appear quite erratic where the frequencies are small (near $x = 0$). This suggests that a smoothing procedure should be used. Note that the results of the binomial and compound binomial are similar, although both are jagged.

The smoothing procedure of Lemon and Krutchkoff [1969] was applied using the $\hat{p}_N(x)$ of Table 3. This procedure essentially obtains estimates by smoothing the $\hat{p}_N(x)$. The estimates appropriate for the binomial distribution are

$$\hat{\mathcal{E}}^*_N(T|x) = \frac{\sum_{i=0}^{n} [\hat{T}_i]^{x+1}[1 - \hat{T}_i]^{N-x}}{\sum_{i=0}^{n} [\hat{T}_i]^{x}[1 - \hat{T}_i]^{N-x}},$$

where $i = 1, \ldots, n$ refers to a summation over the sample and $\hat{T}_i$ is

Table 3

Data from Lord [1965] (Unsmoothed Data) n = 1000

x	$\hat{P}_N(x)$	Compound Binomial Model $\hat{p}_N(x)$	Compound Binomial Model $\mathcal{E}_N(T\|x)$	Compound Binomial Model $E_N(T\|x)$ (a)	Compound Binomial Model $E_N(T\|x)$ (b)	Binomial Model $\hat{\mathcal{E}}(T\|X_N=x)$
0	0.0	.000017	.292	$.64 \times 10^{14}$	2.578	-1.219
1	0.0	.0002	.420	$.70 \times 10^{15}$	5.876	1.382
2	0.001	.0015	.287	.403	.395	.481
3	0.004	.0040	.158	.069	.065	.060
4	0.003	.0053	.368	.476	.467	.526
5	0.012	.0128	.339	.353	.353	.390
6	0.021	.0207	.313	.286	.286	.298
7	0.026	.0267	.349	.360	.360	.372
8	0.035	.0334	.351	.311	.307	.311
9	0.033	.0367	.427	.463	.459	.470
10	0.052	.0471	.429	.408	.405	.411
11	0.044	.0482	.470	.462	.463	.472
12	0.061	.0577	.514	.520	.521	.528
13	0.060	.0617	.538	.518	.519	.523
14	0.072	.0699	.584	.610	.610	.614
15	0.082	.0752	.594	.594	.594	.593
16	0.067	.0684	.597	.566	.565	.559
17	0.059	.0650	.670	.677	.678	.670
18	0.084	.0757	.734	.765	.767	.768
19	0.067	.0690	.714	.681	.682	.675
20	0.070	.0676	.778	.800	.800	.797
21	0.061	.0594	.789	.793	.793	.784
22	0.038	.0417	.797	.790	.790	.774
23	0.035	.0345	.872	.895	.896	.886
24	0.008	.0116	.787	.728	.730	.693
25	0.005	.0060	.939	.969	.969	.957

some estimate of true score for individual i. Following Lemon and Krutchkoff, the initial estimate of $\hat{T}_i$ is $\frac{x_i}{N}$, and an iterated smoothed estimate is obtained by setting $\hat{T}_i$ equal to the initial smoothed estimate, $\hat{\mathcal{E}}^*_N(T|x_i)$. The iterated smoothed estimates are shown in Table 4 along with the corresponding compound binomial estimates using (16) and (17). These estimates appear quite smooth and exhibit monotonicity throughout the range of x.

Table 4

Smoothed Estimates

x	$\mathcal{E}_N^*(T\|x)$	$E_N^*(T\|x)$
0	.218	.192
1	.235	.215
2	.254	.236
3	.273	.258
4	.294	.281
5	.315	.303
6	.338	.327
7	.362	.351
8	.388	.378
9	.416	.408
10	.446	.439
11	.477	.471
12	.508	.504
13	.539	.537
14	.571	.569
15	.602	.602
16	.634	.635
17	.666	.669
18	.698	.702
19	.728	.734
20	.757	.763
21	.783	.790
22	.807	.815
23	.832	.841
24	.860	.870
25	.891	.902

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